A PLANE STEADY HEAT CONDUCTION PROBLEM

Yu. Ya. Iossel and R. A. Pavlovskii

Inzhenerno-Fizicheskii Zhurnal, Vol. 10, No. 2, pp. 258–263, 1966

UDC 536.21

A steady heat conduction problem with mixed conditions assigned at the boundary of a half-space is examined. Two methods of solution are compared, and expressions are given for extreme values of the temperature.

We shall examine the half-space $y > 0$, on the boundary of which mixed boundary conditions of the type (Fig. 1a)

$$T - h\frac{\partial T}{\partial y} = C \text{ when } y = 0,\ |x| < 1,$$

$$T = 0 \text{ when } y = 0,\ |x| > 1 \tag{1}$$

are satisfied.

The first of these boundary conditions corresponds, as is known, to conditions of convective heat transfer, and the second to conditions where the surface is isothermal.

To find the temperature distribution in the half-space $y > 0$ we shall use two different methods, one of which is based on reducing the problem to solution of an integral Fredholm equation, while the other uses the particular properties of the first of the boundary conditions (1) and the method of conformal mapping. A parallel examination of these two methods is warranted by the fact that the first allows us to construct a solution of the problem which is in principle as accurate as we please, while the second leads to a more effective, though admittedly approximate, solution.

Examining the first of the above methods, we shall seek a solution in the usual form:

$$T = \int_0^\infty A(p)\exp(-py)\cos(px)\,dp. \tag{2}$$

Then the boundary conditions (1) reduce to the following system of paired integral equations for the unknown coefficient A(p):

$$\int_0^\infty A(p)(1 + hp)\cos(px)\,dp = C \text{ when } |x| < 1,$$

$$\int_0^\infty A(p)\cos(px)\,dp = 0 \text{ when } |x| > 1. \tag{3}$$

Following [1], we shall seek a solution of this system of equations in the form

$$A(p) = \int_0^1 \varphi'(t)\, I_0(pt)\,dt \tag{4}$$

or (after integration by parts) in the form

$$A(p) = \varphi(1)\, I_0(p) + p\int_0^1 \varphi(t)\, I_1(pt)\,dt. \tag{5}$$

By substituting (4) in the second of Eqs. (3) it is not difficult to verify that it is satisfied identically.

Integrating the first of Eqs. (3) with respect to x, we bring it to the form

$$\int_0^\infty A(p)\sin(px)\,dp + \frac{1}{h}\int_0^\infty \frac{A(p)}{p}\sin(px)\,dp = \frac{Cx}{h}. \tag{6}$$

Substituting (4) and (5), respectively, in the first and second terms of the left side of (6), and using the known expressions [2],

$$\int_0^\infty I_0(pt)\sin(px)\,dp = \begin{cases} 0 \text{ when } 0 < x < t, \\ 1/\sqrt{x^2 - t^2} \text{ when } x > t > 0, \end{cases}$$

$$\int_0^\infty [I_0(p)/p]\sin(px)\,dp = \arcsin x,$$

$$\int_0^\infty I_1(pt)\sin(px)\,dp = \begin{cases} x/t\sqrt{t^2 - x^2} & \text{when } 0 < x < t, \\ 0 & \text{when } x > t > 0, \end{cases}$$

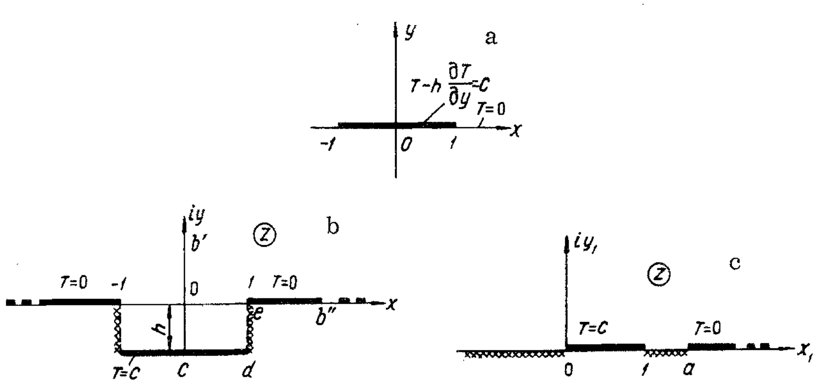

Fig. 1. Diagram of half-space: a) original system; b) approximate model for calculation; c) transformed model for calculation.

we arrive at the equation

$$\int_0^x \frac{\varphi'(t)\,dt}{\sqrt{x^2-t^2}} + \frac{\varphi(1)}{h}\arcsin x + \frac{1}{h}\int_0^1 \psi(x,\,t)\,\varphi(t)\,dt = \frac{Cx}{h}\,^* \tag{7}$$

when $0 < x < 1$,

where

$$\psi(x,\,t) = \begin{cases} x/t\sqrt{t^2-x^2} & \text{when } 0 < x < t, \\ 0 & \text{when } x > t > 0. \end{cases}$$

We now introduce the function

$$f(x) = \int_0^x \frac{\varphi'(t)\,dt}{\sqrt{x^2-t^2}}. \tag{8}$$

Equation (8) may be regarded as an integral Schlömilch equation with respect to function $\varphi'(t)$.

We then have

$$\varphi(t) = \frac{2}{\pi}\int_0^t \frac{sf(s)}{\sqrt{t^2-s^2}}\,ds. \tag{9}$$

Substituting (8) and (9) into (7), we arrive at an integral Fredholm equation of type II relative to function $f(x)$:

$$f(x) + \frac{2}{h\pi}\int_0^1 N(x,\,s)\,f(s)\,ds = \frac{1}{h}\,[Cx - \varphi(1)\arcsin x], \tag{10}$$

where

$$N(x,\,s) = s\int_s^1 \frac{\psi(x,\,t)}{\sqrt{t^2-s^2}}\,dt =$$

$$= \begin{cases} \dfrac{1}{2}\ln\dfrac{x^2-s^2}{x^2+s^2-2x^2s^2-2xs\sqrt{(1-s^2)(1-x^2)}} & (t > x > s), \\[2ex] \dfrac{1}{2}\ln\dfrac{s^2-x^2}{x^2+s^2-2x^2s^2-2xs\sqrt{(1-s^2)(1-x^2)}} & (t > s > x). \end{cases}$$

It is not hard to establish that the kernel $N(x, s)$ is quadratically integrable and symmetrical. Therefore the solution of the problem may in principle be considered found, since the integral equation (10) can be solved by a known method, based, for example, on replacement of the arbitrary kernel by a degenerate one [3]. In the latter case the solution may be obtained with a given degree of accuracy for any values of the parameter $2/h\pi$ which do not coincide with the eigenvalues of (10). As a result, function $f(x)$ is determined to an accuracy within the constant $\varphi(1)$. This constant and the function $\varphi(t)$ are determined from (9), which allows us to find directly the temperature distribution of interest by means of successive use of (4) and (2). Using the method described, however, the solution may be found only numerically, which appreciably restricts the possibility of its interpretation.

We shall therefore examine the possibility of obtaining an analytical expression of the solution, even if admittedly approximate. For this we use the property [4] of characteristic points for boundary conditions of type III. The geometric location of the characteristic points (characteristic surface) for the convective heat transfer surface that we are examining (with h = const) will be an infinitely long strip of unit width, parallel to the plane y = 0 and distance h from it. We assume that the condition $T|_{y=-h} = C$ is satisfied on the characteristic surface. In this case the first of boundary conditions (1) will be satisfied approximately on the surface y = 0, $|x| > 1$; the degree of approximation will be greater, the more linear the temperature variation in the direction of the y axis in the section from the characteristic surface to the surface y = 0, $|x| > 1$, i. e., the greater the ratio of the plate width to the coefficient h (this having the dimension of length, as is easily seen). Thus, the thermal field of interest to us may be found approximately with the aid of the calculation model depicted in Fig. 1b. **

We shall carry out this calculation with the help of conformal mapping. For this purpose we bring into consideration the plane of the complex variable z (Fig. 1b), and conformally map on it the part bounded by the contour bcde,*** this being the upper half-plane of the new complex variable z_1. Then the points of the original plane z = 1; 1 −ih; −ih; ∞ go over, respectively, into the points $z_1 = a; 1; 0; \infty$ of the mapped plane. The desired mapping is given by the Christoffel-Schwarz integral, which in our case leads to the formula

$$z = A\int_0^{z_1} \frac{\sqrt{t-a}}{\sqrt{t(t-1)}}\,dt + B. \tag{11}$$

Because of the conformity of points of the planes z and z_1, we find that the transformation constants A and B are

$$A = 1/2\,\sqrt{a}\,E(\sqrt{1/a}),\quad B = -ih,$$

while the parameter a is determined from the transcendental equation

*Since the solution of the problem examined is an odd function with respect to x, henceforth, for definiteness, we shall put x > 0.

**In constructing this model, we assumed that on the surfaces $|x| = 1$, $-h < y < 0$ the boundary condition $\partial T/\partial x = 0$ is satisfied, which corresponds very closely to the obvious conditions of heat flux distribution in the system being examined.

***Examination of this region proves to be sufficient because of symmetry of the original system.

$$[K'(\alpha) - E'(\alpha)]/E(\alpha) = h, \text{where } \alpha = \sqrt{1/a}.$$

The solution of this equation for any given value of h may easily be found with the aid of tables of elliptic integrals. The corresponding dependence a(h) is shown graphically in Fig. 2.

Thus, the function (11) being mapped has been determined completely, since the integral entering into it has been evaluated for any values of z_1[2]. The next stage of the solution now reduces to finding the thermal field of the system of planes shown in Fig. 1c.

An effective method of solution of similar problems has been put forward by Sochnev [5], and leads to the following expression for the modulus of the temperature gradient q:

$$q = D / \sqrt[4]{(x_1^2 + y_1^2)[(x_1 - 1)^2 + y_1^2][(x_1 - a)^2 + y_1^2]}, \tag{12}$$

where the constant D is determined from the condition

$$\int_1^a q|_{y_1=0}\, dx_1 = C$$

and is equal to

$$D = \sqrt{a}\, C/2K(\sqrt{1 - 1/a}).$$

Expression (12) allows us to find directly the temperature distribution on the surface $x_1 < 0$, which corresponds to the plane of symmetry of the calculation model shown in Fig. 1b:

$$T\Big|_{\substack{x_1<0\\ y_1=0}} = \tag{13}$$

$$= \frac{C}{K(\sqrt{1 - 1/a})} F\left(\arcsin\sqrt{\frac{a}{a - x_1}};\ \sqrt{1 - \frac{1}{a}}\right).$$

Lastly, to determine the error in the solution found, we shall examine the possibility of obtaining upper and lower estimates of the true value of temperature at some arbitrary point of the half-space $y > 0$ (Fig. 1a). For this purpose we shall transfer from the accurate boundary conditions of problem (1) to approximate boundary conditions, one of which may be obtained by equating h = 0 in (1), i. e., having the form

$$T_1 \begin{cases} C \text{ when } |x| < 1,\ y = 0, \\ 0 \text{ when } |x| > 1,\ y = 0. \end{cases} \tag{14}$$

It is evident that the function $T_1(x, y)$ found for these conditions gives an upper estimate for the true temperature distribution $T(x, y)$. At the same time, determination of $T_1(x, y)$ does not present any difficulty and leads to the expression

$$T_1(x, y) = \frac{C}{\pi}\left(\operatorname{arctg}\frac{1 + x}{y} + \operatorname{arctg}\frac{1 - x}{y}\right). \tag{15}$$

In seeking a lower estimate, we go over to approximate boundary conditions of the form

$$T_2 - h\frac{\partial T_1}{\partial y} = C \text{ when } |x| < a_1,\ y = 0,$$

$$T_2 = 0 \text{ when } |x| > a_1,\ y = 0, \tag{16}$$

where T_1 is determined by (15), and the parameter a_1 is found from the condition

$$T_2\Big|_{y=0} = C + h\frac{\partial T_1(a_1, y)}{\partial y}\Big|_{y=0} = 0,$$

which gives

$$a_1 = \sqrt{1 - 2h/\pi}^{\,*}.$$

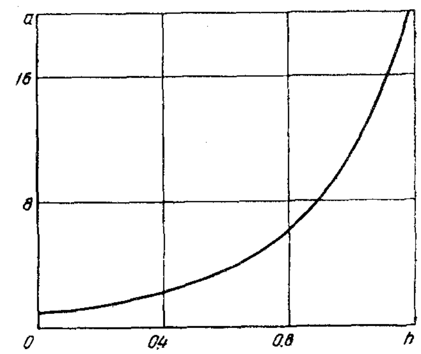

Fig. 2. Graph for determining the transformation parameter.

The need to introduce the parameter a_1 (contraction of the strip examined) is determined by the fact that $T_2\Big|_{\substack{y=0\\ a_1<|x|<1}} < 0$, and the first of the boundary conditions (16) loses its physical meaning when $a_1 < |x| < 1$.

It is easy to show that the function $T_2(x, y)$ constructed in this way actually gives a lower estimate for function $T(x, y)$. To find T_2, we substitute the expression $\frac{\partial T_1}{\partial y}\Big|_{y=0}$, in (16), after which we obtain

$$T_2 = \begin{cases} C[1 - (h/\pi)\cdot 2/(1 - x^2)] & \text{when } |x| < a_1,\ y = 0, \\ 0 & \text{when } |x| > a_1,\ y = 0. \end{cases}$$

An expression for T_2 may easily be obtained with the help of the Poisson integral for the half-plane

$$T_2(x, y) = \frac{C}{\pi}\left(\operatorname{arctg}\frac{a_1 + x}{y} + \operatorname{arctg}\frac{a_1 - x}{y}\right) -$$

$$- \frac{2Chy}{\pi^2[(x^2 + y^2 + 1)^2 - 4x^2]}\left[\frac{x^2 + y^2 + 1}{2}\ln\frac{(1 + a_1)^2}{(1 - a_1)^2} + \right.$$

$$+ x\ln\frac{(x - a_1)^2 + y^2}{(x + a_1)^2 + y^2} - \frac{x^2 - y^2 - 1}{y} \times$$

$$\left. \times \left(\operatorname{arctg}\frac{a_1 + x}{y} + \operatorname{arctg}\frac{a_1 - x}{y}\right)\right]. \tag{17}$$

Thus, expressions (15) and (17) allow us to estimate the accuracy of the solution of the problem examined, both that found from the integral equation (10) and that found from the approximate formula (13).

---

*From this it is clear that the estimate obtained may be used only for values of $h < \pi/2$.

In conclusion, we note that the results obtained may be used not only for the heat conduction problem examined, but also for any problem of potential theory that is similar to it as regards formulation, and in particular to the calculation of the steady electric field of linearly polarized electrodes of appropriate configuration.

NOTATION

T–temperature; T_1, T_2–functions giving upper and lower estimates of the true values of temperature, respectively; h–a quantity which is the reciprocal of the heat transfer coefficient; q–temperature gradient; C–a constant proportional to the temperature of the medium; x, y–rectangular coordinates in the original plane; x_1, y_1–rectangular coordinates in the mapped plane; z, z_1–complex variables; p–integral transformation parameter; s, t–(real) variables of integration; I_0, I_1–Bessel functions of the first kind, of zeroth and first order, respectively; A, a, D–real constants; B–imaginary constant; $K(\alpha)$, $E(\alpha)$–complete elliptic integrals of first and second kind, respectively; α–modulus of elliptic integrals; F–incomplete elliptic integral of first kind.

REFERENCES

1. I. A. Markuzon, PMTF, 5, 69–76, 1963.
2. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, 1962.
3. S. T. Mikhlin, Lectures on Linear Integral Equations [in Russian], Fizmatgiz, 1959.
4. A. V. Lykov, Theory of Heat Conduction [in Russian], Moscow, 1949.
5. A. Ya. Sochnev, Doctoral Dissertation, IAT AN SSSR, 1949.

19 April 1965